EXAM COMPLEX ANALYSIS,
January 29th, 2020, 15:00pm–18:00pm,
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Put your name on every sheet of paper you hand in. Please provide complete arguments for each of your answers. The exam consists of 5 *questions. You can score up to* 9 *points for each question, and you obtain* 5 *points for free.*
In this way you will score in total between 5 *and* 50 *points.*

(1) Recall that the real function arctan sends any $\lambda \in \mathbb{R}$ to the unique η in the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ such that $\tan \eta = \lambda$. Consider $D := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \subset \mathbb{C}$, and define $f\colon D \to \mathbb{C}$ as follows. Let $z \in D$ and write $z = x + iy$ with $x, y \in \mathbb{R}$ and $x > 0$. Then

$$f(z) = (x^2+y^2)^{\frac{1}{4}} e^{\frac{1}{2} i \arctan(y/x)}.$$

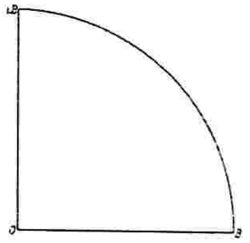
  (a) [3 points.] Use the Cauchy-Riemann relations to show that $f(z)$ is analytic on D.
  (b) [3 points.] Show that $f(z) = \sqrt{z}$ when $z \in D$ is real.
  (c) [3 points.] Use 'analytic continuation' and (b) to draw a conclusion about $f(z)$ for all $z \in D$.

(2) Take $\alpha = \frac{1}{2}\pi + i\log(2) \in \mathbb{C}$ (here $\log(2)$ is simply the classical real natural logarithm), and consider the complex function $f(z) = \frac{\sin z}{z-\alpha}$.
  (a) [3 points.] Explain why $f(z)$ is analytic in every point of $\mathbb{C} \setminus \{\alpha\}$.
  (b) [2 points.] Compute the residue of $f(z)$ in $z = \alpha$.
  (c) [4 points.] Use appropriate closed contours in $\mathbb{C}$ to determine $\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$.

(3) This exercise intends to calculate a certain goniometric integral. Take $f(z) = \frac{1}{2z^2+5iz-2}$.
  (a) [2 points.] Show that $f(z)$ has exactly one pole in the disc given by $|z| < 1$.
  (b) [3 points.] Compute the residue of $f(z)$ in $z = -\frac{1}{2}i$.
  (c) [2 points.] For C the circle parametrized by $t \mapsto e^{it}$ (with $0 \le t \le 2\pi$), show that $\int_C f(z)\,dz = \int_0^{2\pi} \frac{dt}{5+4\sin t}$.
  (d) [2 points.] Calculate $\int_0^{2\pi} \frac{dt}{5+4\sin t}$.

(4) Consider the polynomial $p(z) = z^5 - 10z - 12$
  (a) [3 points.] Show that for $|z| = 1$ one has $|z^5| < |-10z - 12|$.
  (b) [3 points.] Prove that $\int_C \frac{dz}{p(z)} = 0$ where C denotes the closed contour parametrized by $t \mapsto e^{-it}$ (with $0 \le t \le 2\pi$).
  (c) [3 points.] Show that every zero of $p(z)$ satisfies $|z| < 3$.

(5) Take a real constant $B > 1$ and consider the closed contour $\Gamma_B = \alpha_B + \beta_B - \gamma_B$, with $\alpha_B \colon [0, B] \to \mathbb{C}$ given by $\alpha_B(t) = t$ and $\beta_B \colon [0, \frac{1}{2}\pi] \to \mathbb{C}$ given by $\beta_B(t) = e^{it}$ and $\gamma_B \colon [0, B] \to \mathbb{C}$ given by $\gamma_B(t) = it$.



(a) [2 points.] Show that $\int_{\alpha_B - \gamma_B} \frac{dz}{z^4+1} = (1-i)\int_0^B \frac{dt}{t^4+1}$.
(b) [2 points.] Show that the function $\frac{1}{z^4+1}$ has a pole inside the contour Γ_B with residue $-\frac{1}{8}\sqrt{2}\cdot(1+i)$.
(c) [1 point.] Show that if w is a point on β_B, then $\left|\frac{1}{w^4+1}\right| \leq \frac{1}{B^4-1}$.
(d) [2 points.] Prove that $\lim_{B\to\infty} \int_{\beta_B} \frac{dz}{z^4+1} = 0$.
(e) [2 points.] Determine $\int_0^\infty \frac{dt}{t^4+1}$.